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Riemann Curvature Tensor in Nonholonomic Coordinates and Non-Riemannian Space-times

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The calculation of the Riemann Curvature from the deviation of a vector undergoing parallel transport around a closed loop takes a very simple form when expressed in generalized geometrical notation. We discuss the parallel transport of a vector and the type of closed loop used in the calculation. Our method generalizes similar work of Morganstern, to nonholonomic coordinates and non-Riemannian space-times.

1. INTRODUCTION

Central to the understanding of curvature in non-Riemannian space-times, i.e., space-times with independent metric and affine connection, is the concept of autotransport of a vector along a path (Hehl et al., 1976). In a Riemannian space-time this is identical to parallel transport. Curvature can then be interpreted as the measure of the deviation of a vector transported around a closed path. We show here how the coordinate free notation of differential geometry simplifies this calculation and clarifies several misconceptions.

We point out that giving a space-time independent metric and connection is equivalent to giving it torsion (related to the antisymmetric part of the affine connection) and nonmetricity (related to the nonzero covariant derivative of the metric). In what follows we will not need these details, but only the method by which one constructs the covariant derivative.

Morganstern (1977) has correctly pointed out that the standard demonstrations of parallel transport do not include terms that are second order in the differentials, $d\mathbf{u}$. However, the calculation of the curvature obtained by

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the transport of a vector around a closed path is necessarily second order in the differentials. For example, in the transport of the vector $\mathbf{v}$ along the curve described by the tangent vector $\mathbf{u} = d/d\lambda$, the rate of change of $\mathbf{v}$ with respect to λ (called the covariant derivative of $\mathbf{v}$ along $\mathbf{u}$) is given in terms of the transport of $\mathbf{v}(\lambda_0 + \Delta\lambda)$ *back* to λ_0 by

$$\Delta_u \mathbf{v} = \lim_{\Delta\lambda \to 0}\{[\mathbf{v}(\lambda_0 + \Delta\lambda) \text{ transport to } \lambda_0] - \mathbf{v}(\lambda_0)\}/\Delta\lambda \tag{1}$$

[Throughout this paper we will use the geometrical notation of Misner, Thorne, and Wheeler (1973), p. 249.] Equation (1) should be compared with the usual definition (Morganstern, 1977) of the change in a vector v^j when it is transported by an infinitesimal distance $\Delta\lambda$ along a path $u^j(\lambda)$,

$$\delta v^j = -\Gamma_{kl}{}^j v^k u^l \Delta\lambda \tag{2}$$

where $\Gamma_{ij}{}^k$ is the affine connection. The link-up with the geometrical notation (Misner et al., 1973, p. 249) is given by

$$\begin{aligned} \nabla_u \mathbf{v} &= u^j \nabla_{e_j}(v^k \mathbf{e}_k) \equiv u^j [v^k{}_{,j}\mathbf{e}_k + v^k \nabla_{e_j}\mathbf{e}_k] \\ &= u^j [v^k{}_{,j}\mathbf{e}_k + v^k \Gamma_{kj}{}^i \mathbf{e}_i] \\ &= u^j v^k{}_{;j}\mathbf{e}_k \end{aligned} \tag{3}$$

where $\Gamma_{ij}{}^k$ is the affine connection of the spacetime and the semicolon denotes covariant differentiation. This should be compared with the notation of Schouten (1954)

$$\nabla_u v^k = u^j v^k{}_{;j} \tag{4}$$

The action of the two "different" covariant derivatives, ∇_u, should not be confused by the reader. We will, in this article, always mean the former, described by equation (3).

Pictorially, as in Figure 1, we can think of the process described by equation (1) as the expansion of the transported vector [written $\mathbf{v}_{\parallel}(\lambda_0 + \Delta\lambda)$]

$$\begin{aligned} \mathbf{v}_{\parallel}(\lambda_0 + \Delta\lambda) &= \mathbf{v}(\lambda_0) + \nabla_u \mathbf{v}(\lambda_0) \\ &= (1 + \nabla_u)\mathbf{v}(\lambda_0) \end{aligned} \tag{5}$$

Unfortunately this definition of the transport of a vector does not, as shown by Morganstern (1977), give the correct results for the transport of $\mathbf{v}$ along $\mathbf{u}$

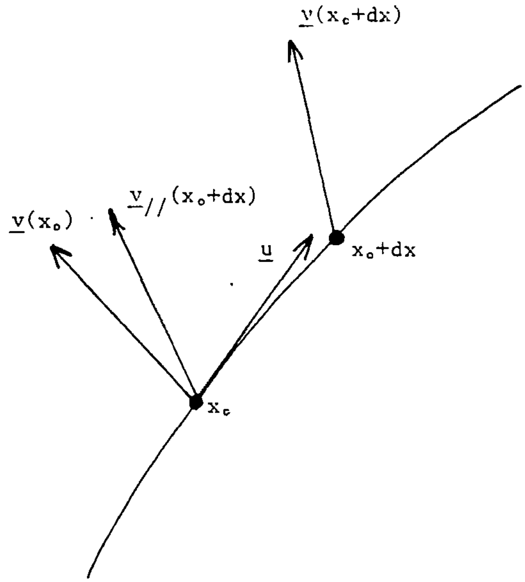

Fig. 1. Parallel transport of the vector $\mathbf{v}(x)$ in the direction $\mathbf{u}$ from the point x_0 to $x_0 + dx$. The transported vector, $\mathbf{v}_{\parallel}(x_0 + dx)$, is the vector $\mathbf{v}(x_0)$ plus the correction called the covariant derivative of $\mathbf{v}$ along $\mathbf{u}$.

and back along $-\mathbf{u}$. We can write the latter as

$$\begin{aligned} \mathbf{v}_{\parallel}(\lambda_1 - \Delta\lambda) &= \mathbf{v}(\lambda_1) + \nabla_{-u}\mathbf{v}(\lambda_1) \\ &= (1 + \nabla_{-u})\mathbf{v}(\lambda_1) \end{aligned} \tag{6}$$

However, $\mathbf{v}(\lambda_1)$ must be the vector obtained by the transport of $\mathbf{v}(\lambda_0)$ to the point $\lambda_1 = \lambda_0 + \Delta\lambda$ given by equation (5), so that for the round trip

$$\begin{aligned} \mathbf{v}_{\parallel}(\lambda_0) = \mathbf{v}_{\parallel}(\lambda_1 - \Delta\lambda) &= (1 + \nabla_{-u})(1 + \nabla_u)\mathbf{v}(\lambda_0) \\ &= (1 - \nabla_u\nabla_u)\mathbf{v}(\lambda_0) \end{aligned} \tag{7}$$

There is a second-order correction which will lead to an erroneous correction to the curvature for a vector transported around a nontrivial area described by two nonparallel vectors. Therefore, the transport law, correct to second order in differentials, should read

$$\mathbf{v}_{\parallel}(\lambda_0 + \Delta\lambda) = \left(1 + \nabla_u + \tfrac{1}{2}\nabla_u\nabla_u\right)\mathbf{v}(\lambda_0) \tag{8}$$

Because of the geometrical notation, this form of the transport law, which is compatible with that of Morganstern (1977), also contains the transport of the covariant derivative along **u**. The round trip calculation described by equation (7) now reads

$$\mathbf{v}_{\parallel}(\lambda_0) = \left(1 + \nabla_{-u} + \tfrac{1}{2}\nabla_{-u}\nabla_{-u}\right)\left(1 + \nabla_u + \tfrac{1}{2}\nabla_u\nabla_u\right)\mathbf{v}(\lambda_0)$$
$$= \left(1 + O\left[(\Delta\lambda)^3\right]\right)\mathbf{v}(\lambda_0) \tag{9}$$

[One could easily surmise that the "exact" form of the parallel transport law is (Schouten, 1954, p. 131)

$$\mathbf{v}(\lambda_0 + \Delta\lambda) = \exp(\nabla_u)\mathbf{v}(\lambda_0) \tag{10}$$

Then equation (9) would yield $v_{\parallel}(\lambda_0) = v(\lambda_0)$ exactly. In what follows, we will only retain terms to second order as discussed above.]

We now apply equation (8) to the transport of a vector around a closed circuit.

2. RIEMANN TENSOR FROM THE TRANSPORT OF A VECTOR

Consider the transport of a vector $\mathbf{v}(x)$ around a closed loop constructed from two nonparallel vector fields **a** and **b** at x_0 as shown in Figure 2. In general, $\mathbf{a}(x_2)$ may be different from the parallel transport of $\mathbf{a}(x_0)$ to the point x_2. These two concepts must be distinguished in the calculation of the Riemann tensor which is obtained from the transport of the vector **v** around a closed loop.

In the former case, the loop is constructed from the vector fields of **a** and **b** and the *coordinate systems which they drag along* (Schouten, 1954; pp. 102–104). Because of the distortion of the geometry, the path formed by $\mathbf{a}(x_0)$ + $\mathbf{b}(x_1)$, where $\mathbf{b}(x_1)$ is the point transformation of **b** along $\mathbf{a}(x_0)$, and $\mathbf{b}(x_0)$ + $\mathbf{a}(x_2)$ may not be close. The vector **c** which closes the curve, called the *closer of quadrilaterals* (Misner et al., 1973; p. 236), is

$$\mathbf{c} = [\mathbf{b}, \mathbf{a}] \tag{11}$$

The vector **c** is also called the Lie derivative of the vector field **a** with respect to the vector field **b**:

$$\pounds_b \mathbf{a} = [\mathbf{b}, \mathbf{a}] \tag{12}$$

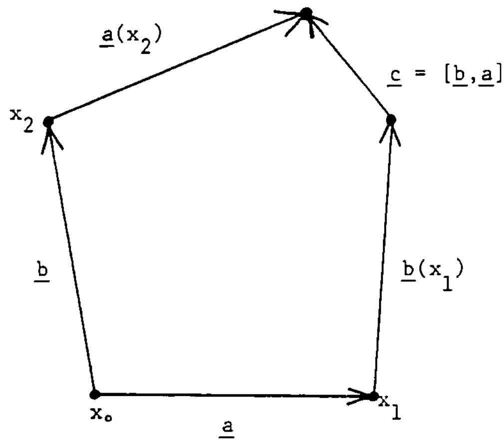

Fig. 2. The parallelogram formed by the vector fields $\mathbf{a}(x_0), \mathbf{b}(x_0), \mathbf{a}(x_2)$ at the tip of $\mathbf{b}(x_0)$, and $\mathbf{b}(x_1)$ at the tip of $\mathbf{a}(x_0)$. The vector $\mathbf{c} = [\mathbf{b},\mathbf{a}]$, is called the closer of quadrilaterals. The process of expanding the fields $\mathbf{a}(x_2)$ and $\mathbf{b}(x_1)$ about the point x_0 to obtain the value of $\mathbf{c}$ is called the dragging of coordinate frames by the vector fields. For holonomic coordinates, $\mathbf{c} = 0$.

For a set of basis vectors $\{\mathbf{e}_i\}$, the commutator takes the form (Misner et al., 1973, p. 239)

$$[\mathbf{e}_i, \mathbf{e}_j] = \begin{cases} 0, & \text{holonomic coordinates} \\ C_{ij}{}^k \mathbf{e}_k, & \text{nonholonomic coordinates} \end{cases} \tag{13}$$

where the tensor $C_{ij}{}^k$ measures the noncommutativity of the basis. The quantity $\Omega_{ji}{}^k = \frac{1}{2} C_{ij}{}^k$ is called *the object of anholonomy* (Schouten, 1954, p. 100) and is used by those who express the basis vectors in terms of tetrads. An excellent and readable introduction to the use of tetrads and their relationship to anholonomic coordinates has been presented by Gogala (1980),[1] who has successfully deciphered the various notations used.

For comparison, we discuss the closed loop obtained from parallel transport. In Figure 3, we note that the parallel transport of the vector $\mathbf{b}(x_1)$ back along $-\mathbf{a}$ to the point x_0 is the vector $\mathbf{b}_{\parallel}(x_0)$ which is located at the point x_1,

$$\mathbf{b}_{\parallel}(x_0) = [1 + \nabla_a]\mathbf{b}(x_1) \tag{14}$$

[1]Note that equation (1.6) for the commutator of two vectors is not valid in the anholonomic frame.

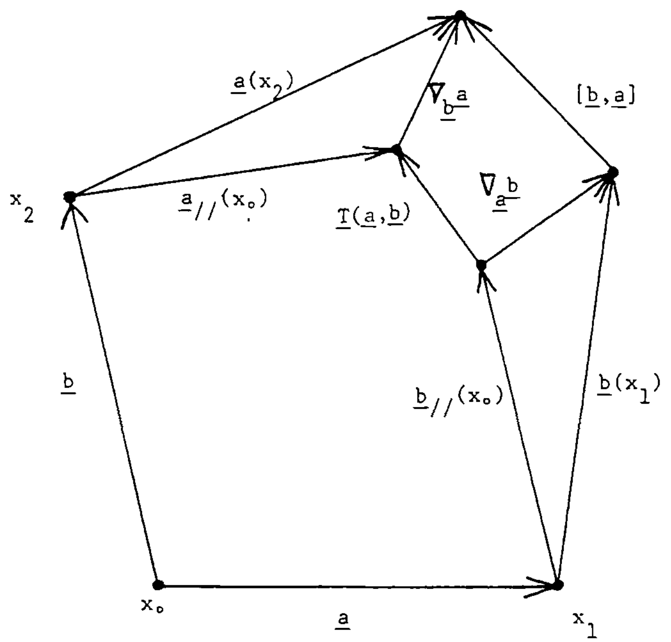

Fig. 3. Nonclosure of quadrilaterals formed by parallel transport compared with dragging of coordinate frames. The parallelogram formed by parallel transport is closed by the torsion vector **T**(**a**, **b**). Note that even in the case of vanishing torsion, the quadrilaterals formed by parallel transport and the dragging of coordinate frames are in general different. They agree only in the case of holonomic coordinates in torsion-free space-times.

where the second-order term has been neglected for the moment. In general, the figure described by two parallelly transported vectors does not close. The vector which measures the difference between two parallelly transported vectors **a** and **b** is called the torsion vector **T**(**a**, **b**) (Hehl et al., 1976). From Figure 3, it is easy to see that the torsion is given by

$$\mathbf{T}(\mathbf{a},\mathbf{b}) = \nabla_a(\mathbf{b}) - \nabla_b(\mathbf{a}) - [\mathbf{a},\mathbf{b}] \tag{15}$$

Thus in a space-time in which the torsion vanishes, the closer of quadrilaterals, constructed either by parallel transport or by the dragging of coordinates (i.e., Lie transport), is identical. However for space-times with torsion this does not happen in general. In this case, the commutator could vanish (holonomic coordinates and zero Lie derivative) so that the parallelogram formed by the vector-field closes; yet, that formed by parallel transport does not.

The question is, which closed loop should one use to calculate the Riemann tensor? We can answer this question by noting that the Riemann curvature tensor is a measure of the tidal force that arises in geodetic deviation between neighboring geodesics. It is the tangent to the geodesics which is transported along the geodesics, across to neighboring geodesics and back. The geodesics are not transported to one another. We then see that the Riemann curvature tensor is obtained from the deviation of a vector parallelly transported around the closed loop built up out of the vector fields and the coordinate systems dragged along. By definition, the Riemann curvature tensor gives the negative of this deviation (Misner et al., 1973, p. 277)

$$-\delta \mathbf{v} = -\left[\mathbf{v}_{\parallel}(x_0) - \mathbf{v}(x_0)\right] = \mathscr{R}(\mathbf{a}, \mathbf{b})\mathbf{v} \tag{16}$$

where $\mathscr{R}(\mathbf{b}, \mathbf{a})$ is the curvature operator. Utilizing equation (8), the transport of the vector $\mathbf{v}$ counterclockwise around the path of Figure 2 gives

$$\begin{aligned}\mathbf{v}_{\parallel}(x_0) &= \left(1 + \nabla_{b+1/2}\nabla_b\nabla_b\right)\left(1 + \nabla_a + \tfrac{1}{2}\nabla_a\nabla_a\right)\left(1 + \nabla_{-[b,a]}\right) \\ &\quad \times \left(1 + \nabla_{-b} + \tfrac{1}{2}\nabla_{-b}\nabla_{-b}\right)\left(1 + \nabla_{-a} + \tfrac{1}{2}\nabla_{-a}\nabla_{-a}\right)\mathbf{v}(x_0) \\ &= \left\{1 + [\nabla_b, \nabla_a] + \nabla_{-[b,a]}\right\}\mathbf{v}(x_0)\end{aligned} \tag{17}$$

Therefore, the curvature operator becomes

$$\mathscr{R}(\mathbf{a}, \mathbf{b}) = [\nabla_a, \nabla_b] - \nabla_{[a,b]} \tag{18}$$

In terms of a set of basis vectors $\{e_i\}$ which may be nonholonomic,

$$\begin{aligned}\mathscr{R}(\mathbf{e}_i, \mathbf{e}_j)\mathbf{e}_k &= \left[\nabla_i, \nabla_j\right]\mathbf{e}_k - \nabla_{[e_i, e_j]}\mathbf{e}_k \\ &= \nabla_i\left(\Gamma^l{}_{kj}\mathbf{e}_l\right) - \nabla_j\left(\Gamma^l{}_{ki}\mathbf{e}_l\right) - C_{ij}{}^l\nabla_l\mathbf{e}_k \\ &= \left(2\Gamma^m{}_{k[j,i]} + 2\Gamma^l{}_{k[j}\Gamma^m_{|l|i]} - C_{ij}{}^l\Gamma^m{}_{kl}\right)\mathbf{e}_m \\ &= R^m{}_{kij}\mathbf{e}_m\end{aligned} \tag{19}$$

which is the standard form of the Riemann curvature tensor.

3. CONCLUSIONS

The simple statement that the Riemann curvature tensor is proportional to the deviation between a vector and the same vector parallelly transferred around a closed loop has been shown to be fraught with complications. This statement is subscribed to by most texts but rarely is the deviation calculated explicitly, as is pointed out by Morganstern (1977), who also showed that most physics texts about relativity do not develop the law of parallel transport with sufficient accuracy to actually carry out the implied calculation. Instead, the calculation that we generally find is the difference between the vector **v** parallelly transported along the path **a**+**b** and then **b**+**a** (see Figure 2). This begs the question of what we are comparing and ignores totally the closure of the quadrilateral, which would have to be put in by hand (Misner et al., 1973, p. 277). Here we differ from Morganstern (1977) in that in nonholonomic coordinates, the closure of the quadrilateral term will yield terms of the order of the area of the quadrilateral that contribute to the Riemann curvature [see equation (19)].

Finally, we argue that the correct path is the quadrilateral constructed from the vector fields and the coordinates which are dragged along, i.e., the expansion of the vectors in terms of the reference point. Combining this closed path with the simple, geometrical description of the parallel transport of a vector, equation (8), we quickly obtain the general form of the curvature which, because of the generalized notation used, is valid for holonomic or anholonomic coordinates in a space-time with or without an independent affine connection.

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